

# A GENERALIZED THEOREM OF KATZ AND MOTIVIC INTEGRATION

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## CONTENTS

Introduction	1
1. Two lemmas	1
2. The theorem	2
3. Application of result	5
References	5

## INTRODUCTION

In what follows, we are interested in an extension of a theorem of Nicholas Katz, which will be useful in studying the cohomology of generalized arc spaces develop by Hans Schoutens in [5] and [6]. As is well known, one is typically interested in the motivic volume of a definable subset of  $\mathcal{X} \times X \times \mathbb{Z}^n$  where  $\mathcal{X}$  is a scheme over  $k((t))$  and  $X$  the special fiber of  $\mathcal{X}$ , cf., [2]. Schoutens has introduced the possibility of developing a motivic integration for *limit points* other than  $k[[t]]$ . In this note, we are concerned with a special type of limit point  $k[[T]]$  where  $T = (t_i)_{i \in \mathbb{N}}$ .

### 1. TWO LEMMAS

We start with a few lemmas from commutative algebra which we will need.

**1.1. Lemma.** *Let  $R := k[[x_1, x_2, \dots, x_n, \dots]]$  be the completion of the polynomial ring  $k[x_1, x_2, \dots, x_n, \dots]$  along the maximal ideal  $\mathfrak{m} = (x_1, x_2, \dots, x_n, \dots)$ . For all  $n \in \mathbb{N}$ , let  $R_n := k[[x_1, \dots, x_n]] \cong R/(x_{n+1}, x_{n+2}, \dots)$  and let  $R_n \rightarrow R_{n-1}$  be the homomorphism with kernel  $(x_n)R_n$ . Then there is an isomorphism*

$$R \cong \varprojlim_n R_n$$

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Moreover,  $R$  is a local ring with maximal ideal  $\mathfrak{m}R$

*Proof.* It is straightforward to verify<sup>1</sup> that  $\mathfrak{m} = \mathfrak{n}R$  is the maximal ideal of  $R$ . For the other claim, we define a homomorphism from  $R$  to  $\varprojlim_n R_n$  by  $x_i \mapsto (y_j)_{j \in \mathbb{N}}$  where  $y_j = 0$  if  $j < i$  and  $y_j = x_i$  if  $i \geq j$ . By definition of inverse limit, this map is injective. Using the universal property of inverse limits, we conclude that it is surjective.  $\square$

Below we will state a version of Nakayama's Lemma which will be important for our work below

**1.2. Lemma.** *Let  $R$  be any local ring (or, in particular, the one above), and let  $M$  be a finitely generated  $R$ -module. Then there is a surjective homomorphism of  $R$ -modules*

$$M/\mathfrak{m}M \otimes_{\mathbb{Z}} R \rightarrow M$$

*Proof.* This is a special case of Proposition 2.6 of [1]  $\square$

## 2. THE THEOREM

From now on, we assume that  $k$  is of characteristic zero. What follows is a natural extension, mutatis mutadis, of a theorem contained in a paper of Katz, cf., Proposition 8.9 of [3]. The original argument is originally due to Cartier, whereas my contribution is to show that it works with an inverse system.

**2.1. Theorem.** *Let  $M$  be finite  $R$ -module with a connection  $\nabla$  arising from the continuous  $k$ -derivations coming from  $R$  to  $M$ . Then  $M^\nabla$  is finitely generated and*

$$M \cong M^\nabla \otimes_k R$$

*Proof.* For all  $i \in \mathbb{N}$  we define

$$D_i = \nabla\left(\frac{\partial}{\partial x_i}\right)$$

and for each  $j \in \mathbb{N}$  we define

$$D_i^{(j)} = \frac{1}{j!} \left( \nabla\left(\frac{\partial}{\partial x_i}\right) \right)^j$$

For any  $n$  and any  $n$ -tuple  $J_n = (j_1, \dots, j_n) \subset \mathbb{N}^n$ , we define the following

$$D^{J_n} = \prod_{i=1}^n D_i^{j_i} \quad x^{J_n} = \prod_{i=1}^n x_i^{j_i} \quad (-1)^{J_n} = \prod_{i=1}^n (-1)^{j_i}$$

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<sup>1</sup>You can do this by verifying that  $\mathfrak{m}$  is the additive subgroup of non-units

Then, for each  $n \in \mathbb{N}$  we successfully define an (additive) endomorphism  $P_n$  by

$$P_n : M \rightarrow M, \quad P_n = \sum_{J_n} (-1)^{J_n} x^{J_n} D^{J_n}$$

Now the action of  $R$  on  $M$  is actually the inverse limit homomorphisms  $\rho_n : R_n \rightarrow M$  of  $k$ -modules. In fact,  $P_n$  will be considered an additive endomorphism of  $M$  as an  $R_n$ -module (via the isomorphism established in Proposition 1). We define  $P : M \rightarrow M$  to be the inverse limit

$$P = \varprojlim_n P_n$$

More explicitly, consider  $f \in R$ , which by Proposition 1, can be identified with a sequence  $(f_n)_{n \in \mathbb{N}}$  where  $f_n \in R_n$ , then

$$P(fm) = (P_n(f_n))P(m)$$

It is straightforward, that  $P_n(f_n) = f_n(0) \forall n \in \mathbb{N}$ , from which it follows that for all  $f \in R$  and all  $m \in M$

$$P(fm) = f(0)P(m)$$

Therefore, the kernel of  $P$  contains  $\mathfrak{m}M$  where  $\mathfrak{m}$  is the maximal ideal of  $R$ .

As we will now pass to the quotient  $M/\mathfrak{m}M$ , we mention that it is not hard to see that the inverse system defined in Proposition 1 and hence above satisfies the Mittag Leffler Condition. Therefore, there is an isomorphism

$$M/\mathfrak{m}M \cong \varprojlim_n M/(x_1, \dots, x_n)M$$

Note that, for all  $n$ ,  $P_n$  induces the identity on  $M/(x_1, \dots, x_n)M$ , and so  $P$  induces the identity on  $M/\mathfrak{m}M$  – i.e.,

$$P(m) \equiv m \pmod{\mathfrak{m}}$$

Therefore, the kernel of  $P$  is  $\mathfrak{m}$ . In a similar fashion it is easy to check that  $P$  as the following properties

$$P|_{M^\nabla} = id_{M^\nabla} \quad P(M) \subset M^\nabla \quad P^2 = P$$

Therefore,  $P$  induces an isomorphism vector spaces over  $k$

$$M/\mathfrak{m}M \cong M^\nabla$$

Therefore,  $M^\nabla$  is a finite  $R$ -module. Using Nakayama's Lemma (see Proposition 2 above), we have a surjective map

$$M^\nabla \otimes_k R \rightarrow M$$

Now, we will show that it is an isomorphism. Let  $m_1, \dots, m_l$  be  $k$ -linearly independent elements of  $M^\nabla$  and let  $f_1, \dots, f_l$  be element of  $R$ , we need to show

$$\sum_{k=1}^l f_k m_k \neq 0$$

In other words, writing  $f_k$  as its corresponding sequence  $(f_n^{(k)})$  in the inverse system, we need to show that for sufficiently large  $n$

$$\sum_{k=1}^l f_n^{(k)} m_k \neq 0 \quad (*)$$

The only reason we specify that  $n$  be sufficiently large is to insure that there is an  $n$  so that  $f_n^{(k)} \neq 0$  for some  $k$ , which is clearly satisfied or else there is nothing to prove. Thus, we may assume there exists an  $N$  such that for all  $n > N$

$$f_n^{(1)} \neq 0$$

Then for all  $n \geq N$  there exists a  $n$  tuple  $J_n = (j_1, \dots, j_n)$  such that

$$\Pi_{\nu=1}^n \frac{1}{j_\nu!} \left( \frac{\partial}{\partial x_\nu} \right)^{j_\nu} (f_n^{(1)})(0) \neq 0$$

Now, assume for the sake of contradiction that

$$\sum_{k=1}^l f_n^{(k)} m_k = 0$$

Applying  $D^{J_n}$  to this equation, we get

$$0 = D^{J_n} \left( \sum_{k=1}^l f_n^{(k)} m_k \right) = \sum_{k=1}^l \Pi_{\nu=1}^n \frac{1}{j_\nu!} \left( \frac{\partial}{\partial x_\nu} \right)^{j_\nu} (f_n^{(i)}) m_k$$

This is a sum of the form

$$\sum_{k=1}^l g_k m_k = 0, \quad g_1(0) \neq 0, \quad g_k \in R_n$$

Applying  $P$  to this sum, we obtain

$$\sum_{k=1}^l g_k(0) m_k = 0$$

which is impossible as  $g_1(0) \neq 0$  and the  $m_1, \dots, m_l$  are a  $k$ -linearly independent set. Therefore, this must be an isomorphism.  $\square$

### 3. APPLICATION OF RESULT

To apply the above theorem, we only need to take  $M = H_{DR}(X/S)$  to be finite sheaf of modules on  $S$ , which is assured to us when we take  $f : X \rightarrow S$  to be locally of finite type. We can define arc spaces by a universal property: we say that  $T \rightarrow X$  is the arc space of  $X$  along a scheme  $Z$ , working in the category of  $k$ -schemes, if for every closed fat point  $\eta$  of  $T$  we have a unique morphism  $\eta \times_k Z \rightarrow X$  making the following diagram commute

$$\begin{array}{ccc} \eta & \longrightarrow & \eta \times_k Z \\ \downarrow & & \downarrow \\ T & \longrightarrow & X \end{array}$$

and which is unique in the sense that if  $T' \rightarrow X$  is any other such space, we have a unique map  $T' \rightarrow T$ . When such a scheme exists, we write  $\nabla_Z X$  for the arc space of  $X$  along  $Z$ . This is a generalization of the notion of arc space found in [6].

Using this description of  $\nabla_{\mathfrak{m}} X$  we can conclude that  $\nabla_{\mathfrak{m}} X$  is a scheme over  $\mathfrak{m}$ . In particular, if  $\mathfrak{x}$  is a limit point, then we have the following relation

$$H_{DR}(\nabla_{\mathfrak{x}} X/\mathfrak{x})^\nabla \cong H_{DR}(X/k)$$

when  $\nabla_{\mathfrak{x}} X \rightarrow \mathfrak{x}$  is smooth. This last condition implies, for suitable point systems, that  $X$  is rationally  $\mathfrak{x}$ -laxly stable – cf., [4]. Therefore, we would expect a further decomposition of  $H_{DR}(\nabla_{\mathfrak{x}} X/\mathfrak{x})^\nabla$  which is captured motivically by the rational motivic measure as displayed loc. cit.

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